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ASYMPTOTIC SOLUTION OF THE EQUATIONS OF MOTION FOR A CELTIC STONE*

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An attempt is made to produce a theoretical explanation of the fact that transverse oscillations of a celtic stone are transformed into rotation. The equations of motion are considered in the neighborhood of the position of stable equilibrium, and they contain only the linear and quadratic terms relative to initial perturbations, the latter assumed to be small. The method of averaging is used to integrate the resulting system. A concrete model the numerical values of the parameters of which are given in /1/, is used for illustration.

The celtic stone, also called a Magnus body, is a top with nonsymmetrical distribution of mass such, that the stability of its rotation about the vertical depends on the direction of rotation. It is also asserted that if such a solid is brought into an unstable rotation, then after a short interval the rotation will cease, the body will begin to oscillate about the horizontal axis, and then resume the rotation in the opposite direction. In certain cases this may be repeated a large number of times.

Let the top move along a fixed horizontal plane in such a manner, that the velocity of slippage of the point of contact of the body with the plane is equal zero. It follows that a nonholonomic constraint is imposed on the motion of the top. The problem was first studied by Walker in /2/ and Magnus in /3,4/. The first systematic study of the stability of rotations carried out by V.V. Rumiantsev in /5/ and supplemented by the authors of /6-8/ made it possible, in particular, to explain the dependence of the stability of rotations of the body on the direction of rotation. The phenomenon was observed by Walker /9/ who made various models of the celtic stone. In another experiment, a stationary top was dealt a blow to its upper part. This produced an oscillation about the horizontal axis which decayed rapidly, and transformed itself into a rotation about the vertical, its direction depending on the position of the part struck. A mathematical model attempting to explain these phenomena was constructed in /10/, although in the author's opinion the model could not be related to any real physical model in spite of possessing the properties of the celtic stone. Another quantitative study of the equations of motion for a model of the celtic stone carried out in /1/ describes, with great clarity, all phenomena discussed in /9/.

1. Equations of motion /6/.

The celtic stone S moves without slippage on a horizontal plane π (Fig.1). The position of the body S of mass m is described by the coordinates x_0, y_0 on the horizontal plane of its center of inertia G relative to the fixed trihedron $O_0x_0y_0z_0$ (the plane $x_0O_0y_0$ coincides with the plane π) and Euler angles θ, ψ, φ determining the orientation of the coordinate system $Gx_1x_2x_3$ the axes of which are directed along the principal central axes of inertia of the body S relative to the fixed trihedron. The Lagrangian of the system and conditions of rocking without slippage have the form /6/

$$L = \frac{m}{2}\{x_0^{\cdot 2} + y_0^{\cdot 2} + [\gamma_2\varphi^{\cdot}\sin\theta + \theta^{\cdot}(\gamma_1\cos\theta - \zeta\sin\theta)]^2\} +$$
$$\frac{1}{2}[A(\theta^{\cdot}\cos\varphi + \psi^{\cdot}\sin\theta\sin\varphi)^2 + B(\theta^{\cdot}\sin\varphi - \psi^{\cdot}\sin\theta\cos\varphi)^2 +$$
$$C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta)^2]$$

$$x_0^{\cdot} = \alpha_1\theta^{\cdot} + \alpha_2\varphi^{\cdot} + \alpha_3\psi^{\cdot}, \; y_0^{\cdot} = \beta_1\theta^{\cdot} + \beta_2\varphi^{\cdot} + \beta_3\psi^{\cdot}$$
$$\alpha_1 = -\sin\psi(\gamma_1\sin\theta + \zeta\cos\theta), \; \gamma_1 = \xi\sin\varphi + \eta\cos\varphi$$
$$\alpha_2 = \gamma_1\cos\psi + \gamma_2\cos\theta\sin\psi, \; \gamma_2 = \xi\cos\varphi - \eta\sin\varphi$$
$$\alpha_3 = (\gamma_1\cos\theta - \zeta\sin\theta)\cos\psi + \gamma_2\sin\psi, \; \beta_i = -\partial\alpha_i/\partial\psi$$
$$(i = 1, 2, 3)$$

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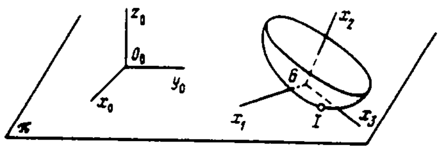

Fig.1

Here A, B and C are the principal moments of inertia of the body S, ξ, η, ζ are the coordinates of the point I of contact between the body S and plane π in the $Gx_1x_2x_3$ coordinate system, the latter being functions of θ and φ determined by the equations

$$\xi = \frac{1}{\Delta}\left(Q\frac{\operatorname{ctg}\theta}{\cos\varphi} - R\operatorname{tg}\varphi\right), \quad \zeta = \frac{1}{\Delta}\left(Q\operatorname{tg}\varphi - P\frac{\operatorname{ctg}\theta}{\cos\varphi}\right)$$
$$\eta = -a + \frac{1}{2}(P\xi^2 + 2Q\xi\zeta + R\zeta^2)$$

where a is the distance between the points G and I, while P, Q, R and Δ are constants connected with the principal radii of curvature ρ_1 and ρ_2 of the outer surface of the body S at the point I, by the relations

$$P = \frac{\cos^2\alpha}{\rho_1} + \frac{\sin^2\alpha}{\rho_2}, \quad Q = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos\alpha\sin\alpha$$
$$R = \frac{\sin^2\alpha}{\rho_1} + \frac{\cos^2\alpha}{\rho_2}, \quad \Delta = \frac{1}{\rho_1\rho_2}$$

The angle α determines the position of the principal axes of curvature of the point I relative to the axes Gx_1, Gx_3. We further assume that $\rho_2 > \rho_1$, $0 < \alpha < \pi/2$, $A > C$. The system in question is conservative and represents a nonholonomic Chaplygin system. The equations of motion are not given here because of their bulk (see e.g. /7/). The author of /6/ has shown the existence of a family of particular solutions

$$\theta = \pi/2, \ \varphi = 0, \ \psi^{\cdot} = \omega \tag{1.1}$$

where ω is an arbitrary constant. The solutions correspond to a uniform rotation of the body S about the vertical axis Gx_2.

2. Stability of rotation. To study the stability of motion (1.1), we introduce three variables $u = \theta - \pi/2$, $v = \varphi$, $x^{\cdot} = \psi^{\cdot} - \omega$. The equations of perturbed motion are written in the form

$$\bar{A}u^{\cdot\cdot} + K_{11}u + K_{12}v - \omega\lambda_1 u^{\cdot} - \omega\lambda_2 v^{\cdot} + U = 0 \tag{2.1}$$
$$\bar{C}v^{\cdot\cdot} + K_{12}u + K_{22}v + \omega\lambda_1 v^{\cdot} - \omega\mu_1 u^{\cdot} + V = 0, \ Bx^{\cdot\cdot} + W = 0$$
$$\bar{A} = A + ma^2, \quad \bar{C} = C + ma^2, \quad K_{11} = \omega^2\left(B - \bar{C} + ma\frac{P}{\Delta}\right) + mg\left(\frac{P}{\Delta} - a\right)$$
$$K_{12} = \frac{mQ}{\Delta}(g + a\omega^2), \quad K_{22} = \omega^2\left(B - \bar{A} + ma\frac{R}{\Delta}\right) + mg\left(\frac{R}{\Delta} - a\right)$$
$$\lambda_1 = ma\frac{Q}{\Delta}, \quad \lambda_2 = B - \bar{A} - \bar{C} + ma\frac{R}{\Delta}$$
$$\mu_1 = \bar{A} + \bar{C} - B - ma\frac{P}{\Delta}$$

where U, V, W are functions of $u^{\cdot}, v^{\cdot}, u, v, x^{\cdot}$ the expansions of which contain no linear terms with respect to the variables given. The characteristic equation of the linear system obtained from (2.1) has a double zero root, and other roots are given by the equation

$$D(s) \equiv a_0 s^4 + a_1\omega s^3 + a_2 s^2 + a_1\omega^3 s + a_3 = 0$$
$$a_0 = \bar{A}\bar{C}, \ a_1 = \lambda_1(\bar{A} - \bar{C}), \ a_2 = \bar{A}K_{22} + \bar{C}K_{11} - \omega^2(\lambda_1^2 + \lambda_2\mu_1)$$
$$a_3 = K_{11}K_{22} - K_{12}^2$$

It has been shown in /11/ that here we have a particular case of the Malkin theorem /12/ and this implies the following sufficient conditions of stability:

$$a_1\omega > 0, \ a_3 > 0, \ -a_0\omega^4 + a_2\omega^2 - a_3 > 0 \tag{2.2}$$

If one of the above inequalities does not hold, then at least one of the roots $D(s) = 0$ will have a positive or zero real part. On the other hand, $s = i\rho$ (ρ is real) can be a root of the equation $D(s) = 0$ only if

$$\rho = \omega, \ -a_0\omega^4 + a_2\omega^2 - a_3 = 0 \tag{2.3}$$

When condition (2.3) holds, we can write $D(s)$ in the form

$$D(s) \equiv (s^2 + \omega^2)D_1(s), \ D_1(s) \equiv a_0 s^2 + \omega a_1 s + a_2 - a_0\omega^2$$

When a quantity $a_1\omega$ is not zero, the equation $D_1(s) = 0$ has no purely imaginary roots. This leads us to conclusion that if at least one of the following inequalities holds:

$$a_1\omega < 0, \ a_3 < 0, \ -a_0\omega^4 + a_2\omega^2 - a_3 < 0$$

then we have instability. In order to interpret the conditions (2.2), we shall first study the case of equilibrium $(\omega = 0)$, where we have the following biquadratic equation:

$$D(s) \equiv a_0 s^4 + a_2^0 s^2 + a_3^0 = 0$$

$$a_2^0 = mg\,[\bar{A}(\rho_1 \cos^2\alpha + \rho_2 \sin^2\alpha - a) + \bar{C}(\rho_1 \sin^2\alpha + \rho_2 \cos^2\alpha - a)]$$

$$a_3^0 = m^2 g^2 (\rho_1 - a)(\rho_2 - a)$$

This yields the necessary conditions for the stability of the equilibrium

$$a_2^0 > 0, \ a_2^{0*} - 4a_0 a_3^0 > 0, \ a_3^0 > 0$$

and in particular the condition

$$(\rho_1 - a)(\rho_2 - a) > 0 \tag{2.4}$$

is a necessary condition of equilibrium.

It can be shown that /5/ in the case of equilibrium the energy integral enables us to construct a Liapunov function. We obtain the following sufficient conditions of equilibrium:

$$\rho_1 \sin^2\alpha + \rho_2 \cos^2\alpha - a > 0, \ (\rho_1 - a)(\rho_2 - a) > 0$$

Combining these conditions with (2.4) we find that if

$$a < \rho_1 < \rho_2 \tag{2.5}$$

then the equilibrium is stable, and unstable if $a > \rho_1$. In what follows, we shall assume that condition (2.5) holds.

Let us now consider the general case of $\omega \neq 0$. The conditions (2.2) are now written

$$(\rho_2 - \rho_1)(\bar{A} - \bar{C})\,\omega \sin 2\alpha > 0 \tag{2.6}$$

$$l_1\omega^4 + l_2\omega^2 + a_3^\circ > 0, \quad l_3\omega^2 - \frac{a_3^\circ}{mg} > 0 \tag{2.7}$$

$$l_1 = (B - \bar{A})(B - \bar{C}) + \frac{ma}{2}[(\rho_1 + \rho_2)(2B - \bar{A} - \bar{C}) + (\rho_2 - \rho_1)(\bar{A} - \bar{C})\cos 2\alpha] + m^2 a^2 \rho_1 \rho_2$$

$$l_2 = \frac{ma}{2}[(\rho_1 + \rho_2 - 2a)(2B - \bar{A} - \bar{C}) + (\rho_2 - \rho_1)(\bar{C} - \bar{A}) + 4(\rho_1\rho_2 - a^2)]$$

$$l_3 = (\bar{A} + \bar{C} - B)(\rho_1 + \rho_2 - 2a) + ma(a\rho_1 + a\rho_2 - 2\rho_1\rho_2)$$

The inequality (2.6) can be reduced to the condition $\omega > 0$, therefore the stability depends on the direction in which the body rotates. If on the other hand $\omega < 0$, then we have instability. If $\omega > 0$, then we have stability, provided that ω^2 satisfies the inequalities (2.7).

Let the condition (2.5) hold and the quantity a_3° be positive. Then the second inequality of (2.7) holds only when the following two conditions hold:

$$l_3 > 0, \quad \omega^2 > \frac{a_3^\circ}{mgl_3}$$

The first inequality imposes a constraint on the form and mass distribution of the body S, and is written in the form

$$\bar{A} + \bar{C} - B > ma\frac{2\rho_1\rho_2 - a\rho_1 - a\rho_2}{\rho_1 + \rho_2 - 2a} \tag{2.8}$$

The second inequality imposes a constraint on the angular velocity of rotation of the body S. The velocity must exceed the critical velocity ω_0 given by the equation

$$\omega_0^2 = \frac{mg(\rho_1 - a)(\rho_2 - a)}{(\bar{A} + \bar{C} - B)(\rho_1 + \rho_2 - 2a) + ma(a\rho_1 + a\rho_2 - 2\rho_1\rho_2)}$$

The above analysis implies that the stability of rotation of the body S depends on the direction of rotation. Particular models of the celtic stone exist, which are unstable irrespective of the direction and angular velocity of the motion. These are the models for which condition (2.8) does not hold. In other cases when the inequality (2.8) holds, we can still have instability in either direction of motion if the initial angular velocity is not sufficiently great.

Interpretation of the first inequality of (2.7) becomes more difficult in the general case when the principal moments of inertia of the body S are arbitrary. However, in the majority of the celtic stone models under consideration we assume, that the axis of vertical rotation coinciding, for some family of motions with the principal axis of inertia, is the axis for which the moment of inertia is greatest. For this reason we shall assume, from now on, that $B > \bar{A} > \bar{C}$, which implies that a quantity l_1 will always be positive. It can be shown that l_2 is positive when $m\rho_1\rho_2 > B$. This condition holds for the model discussed in /1/, and the first inequality of (2.7) holds for such a body at any rate of rotation.

3. Oscillations near the position of equilibrium. ($\theta = \pi/2$, $\varphi = \psi^{\cdot} = 0$). Before investigating the effect which the nonlinearities have on the motion of the body S, we shall write the equations of perturbed motion near the position of equilibrium, with the second order terms included in the explicit form. The system (2.1) now becomes

$$\bar{A}u^{\cdot\cdot} + H_{11}u + H_{12}v - x^{\cdot}(\lambda_1 u^{\cdot} + \lambda_2 v^{\cdot}) + U_1 = 0 \tag{3.1}$$

$$\bar{C}v^{\cdot\cdot} + H_{12}u + H_{22}v + x^{\cdot}(-\mu_1 u^{\cdot} + \lambda_1 v^{\cdot}) + V_1 = 0$$

$$Bx^{\cdot\cdot} - k_1 u^{\cdot} v^{\cdot} - k_2 u^2 - k_3 uv - k_4 v^2 + W_1 = 0$$

$$H_{11} = mg\left(\frac{P}{\Delta} - a\right), \quad H_{12} = mg\frac{Q}{\Delta}, \quad H_{22} = mg\left(\frac{R}{\Delta} - a\right)$$

$$k_1 = \bar{C} - \bar{A} + B, \quad k_2 = mg\frac{Q}{\Delta}\left[-\frac{A}{\bar{A}} + ma\frac{P}{\Delta}\left(\frac{\bar{A} - \bar{C}}{\bar{A}\bar{C}}\right)\right]$$

$$k_3 = mg\left[a\frac{B}{\bar{A}} - \frac{RA}{\Delta\bar{A}} + \frac{P}{\Delta}\left(\frac{C}{\bar{C}} - \frac{B}{\bar{A}}\right) + ma\left(\frac{PR + Q^2}{\Delta^2}\right)\left(\frac{\bar{A} - \bar{C}}{\bar{A}\bar{C}}\right)\right]$$

$$k_4 = mg\frac{Q}{\Delta}\left[\frac{C}{\bar{C}} - \frac{B}{A} + ma\frac{R}{\Delta}\left(\frac{\bar{A} - \bar{C}}{\bar{A}\bar{C}}\right)\right]$$

Here U_1, V_1, W_1 are functions of $u^{\cdot}, v^{\cdot}, u, v, x^{\cdot}$ the expansions of which into series begin with terms of at least third order in these variables. Let the perturbations $u_0^{\cdot}, v_0^{\cdot}, u_0, v_0, x_0^{\cdot}$ be small at the initial instant, and or order ε ($\varepsilon > 0$ is a small parameter). Since the inequalities (2.5) hold, it follows that the equilibrium is stable; in the perturbed motion $u^{\cdot}, v^{\cdot}, u, v, x^{\cdot}$ are infinitesimals of the order of ε.

Using the method of averaging, we shall seek an approximate solution to the system (3.1) in the form of an expansion asymptotic in ε. Following the method given in /10/, we carry out a consecutive change of variables so as to reduce the system (3.1) to its standard form. We write

$$y = u\sqrt{\bar{A}}, \quad z = v/\sqrt{\bar{C}} \tag{3.2}$$

Then the linearized system of equations obtained from (3.1) can be written in the form

$$y^{\cdot\cdot} + E_1 y + E_3 z = 0, \quad z^{\cdot\cdot} + E_3 y + E_2 z = 0, \quad Bx^{\cdot\cdot} = 0$$

$$E_1 = \frac{H_{11}}{\bar{A}}, \quad E_2 = \frac{H_{22}}{\bar{C}}, \quad E_3 = \frac{H_{12}}{\sqrt{\bar{A}\bar{C}}}$$

In addition to the zero root, the corresponding characteristic equation has roots which can be found from the biquadratic equation

$$D(s) \equiv (s^2 + E_1)(s^2 + E_2) - E_3^2 = 0$$

Since the equilibrium is stable, the roots $D(s) = 0$ are purely imaginary: $\pm ib$ and $\pm id$, where b and d are real and positive. Let us assume that, e.g. $b > d$. We introduce the following symmetric matrix:

$$E = \left\| \begin{matrix} E_1 & E_3 \\ E_3 & E_2 \end{matrix} \right\|$$

where E is a positive definite matrix with eigenvalues b^2 and d^2. Let $T(\gamma)$ be a rotation matrix such that

$$T(-\gamma)ET(\gamma) = \left\| \begin{matrix} b^2 & 0 \\ 0 & d^2 \end{matrix} \right\|, \quad T(\gamma) = \left\| \begin{matrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{matrix} \right\|$$

Defining the new variables Y and Z thus

$$\left\| \begin{matrix} Y \\ Z \end{matrix} \right\| = T(-\gamma) \left\| \begin{matrix} y \\ z \end{matrix} \right\|$$

and taking (3.2) into account, we transform the system (3.1) to the form

$$Y^{\cdot\cdot} + b^2Y = x^{\cdot}(N_1Y^{\cdot} + N_2Z^{\cdot}) + U_2,\ Z^{\cdot\cdot} + d^2Z = x^{\cdot}(N_3Y^{\cdot} + N_4Z^{\cdot}) + V_2 \tag{3.3}$$
$$Bx^{\cdot\cdot} = h_1(Y^{\cdot 2} - Z^{\cdot 2}) + h_2Y^{\cdot}Z^{\cdot} + K_2Y^2 + K_3YZ + K_4Z^2 + W_2$$

The quantities U_2, V_2, W_2 are functions of $Y^{\cdot}, Z^{\cdot}, Y^{\cdot}, Z, x^{\cdot}$, and their expansions into series contain terms of at least third order. The constants $N_1, \ldots, K_4$ are given by the equalities

$$N_1 = \bar{\lambda}_{1a}\cos^2\gamma + (\bar{\lambda}_2 + \bar{\mu}_1)\sin\gamma\cos\gamma - \bar{\lambda}_{1c}\sin^2\gamma$$
$$N_2 = \bar{\lambda}_2\cos^2\gamma\,(\bar{\lambda}_{1a} + \bar{\lambda}_{1c})\sin\gamma\cos\gamma - \bar{\mu}_1\sin^2\gamma$$
$$N_3 = -\bar{\lambda}_2\sin^2\gamma - (\bar{\lambda}_{1a} + \bar{\lambda}_{1c})\sin\gamma\cos\gamma + \bar{\mu}_1\cos^2\gamma$$
$$N_4 = \bar{\lambda}_{1a}\sin^2\gamma - (\bar{\lambda}_2 + \mu_1)\sin\gamma\cos\gamma - \bar{\lambda}_{1c}\cos^2\gamma$$
$$h_1 = \bar{k}_1\sin\gamma\cos\gamma,\ h_2 = \bar{k}_1\cos 2\gamma,\ K_2 = \bar{k}_2\cos^2\gamma + \bar{k}_3\sin\gamma\cos\gamma + \bar{k}_4\sin^2\gamma$$
$$K_3 = (\bar{k}_4 - \bar{k}_2)\sin 2\gamma + \bar{k}_3\cos 2\gamma,\ K_4 = \bar{k}_2\sin^2\gamma - \bar{k}_3\sin\gamma\cos\gamma + \bar{k}_4\cos^2\gamma$$
$$\bar{\lambda}_{1a} = \frac{\lambda_1}{A},\quad \bar{\lambda}_{1c} = \frac{\lambda_1}{C},\quad \bar{\lambda}_2 = \frac{\lambda_2}{\sqrt{AC}},\quad \bar{\mu}_1 = \frac{\mu_1}{\sqrt{AC}}$$
$$\bar{k}_1 = \frac{k_1}{\sqrt{AC}},\quad \bar{k}_2 = \frac{k_2}{A},\quad \bar{k}_3 = \frac{k_3}{\sqrt{AC}},\quad \bar{k}_4 = \frac{k_4}{C}$$

The variables $Y^{\cdot}, Z^{\cdot}, Y, Z, x^{\cdot}$ are infinitesimals of the order of ε, i.e. lf the order of initial perturbations. Therefore, we seek a solution of (3.3) in the form

$$Y = \varepsilon\bar{Y},\ Z = \varepsilon\bar{Z},\ x^{\cdot} = \varepsilon\bar{x}^{\cdot}$$

where $\bar{Y}, \bar{Z}$ and $\bar{x}^{\cdot}$ are found from the system

$$\bar{Y}^{\cdot\cdot} + b^2\bar{Y} = \varepsilon F_1,\quad \bar{Z}^{\cdot\cdot} + d^2\bar{Z} = \varepsilon F_2,\quad B\bar{x}^{\cdot\cdot} = \varepsilon F_3$$
$$F_1 = \bar{x}^{\cdot}(N_1\bar{Y}^{\cdot} + N_2\bar{Z}^{\cdot}) + o(\varepsilon),\quad F_2 = \bar{x}^{\cdot}(N_3\bar{Y}^{\cdot} + N_4\bar{Z}^{\cdot}) + o(\varepsilon)$$
$$F_3 = h_1(\bar{Y}^{\cdot 2} - \bar{Z}^{\cdot 2}) + h_2\bar{Y}^{\cdot}\bar{Z}^{\cdot} + K_2\bar{Y}^2 + K_3\bar{Y}\bar{Z} + K_4\bar{Z}^2 + o(\varepsilon)$$

Introducing new variables $A_1, A_2, \theta_1, \theta_2$ defined by the equations

$$\bar{Y} = A_4\cos(bt + \theta_1),\ \bar{Z} = A_2\cos(dt + \theta_2)$$
$$\bar{Y}^{\cdot} = -A_1b\sin(bt + \theta_1),\ \bar{Z}^{\cdot} = -A_2\,d\sin(dt + \theta_2)$$

we can wirte the system (3.3) in the following matrix form:

$$X^{\cdot} = \varepsilon f(X, t, \varepsilon) \tag{3.4}$$

where (T denotes transposition)

$$X^T = [A_1, \theta_1, A_2, \theta_2, \bar{x}^{\cdot}],\ f^T = [f_1, f_2, f_3, f_4, f_5]$$
$$f_1 = -\frac{1}{b}F_1\sin(bt + \theta_1),\quad f_2 = -\frac{1}{A_1b}F_1\cos(bt + \theta_1)$$
$$f_3 = -\frac{1}{d}F_2\sin(dt + \theta_2),\quad f_4 = -\frac{1}{A_2d}F_2\cos(dt + \theta_2),\quad f_5 = \frac{F_3}{B}$$

The method of averaging enables us to obtain /13/ an approximate solution to the system (3.3) for small ε. Indeed, the function f is almost periodic in t, therefore we can define the function

$$\langle f(x)\rangle = \lim_{t\to\infty}\left[\frac{1}{t}\int_0^t f(x, u, 0)\,du\right]$$

The solutions of the averaged system

$$x^{\cdot} = \varepsilon\langle f(x)\rangle \tag{3.5}$$

make it possible to obtain approximate solutions of the system (3.4). The system (3.5) has the form

$$A_1^{\cdot} = \frac{\varepsilon}{2}N_1A_1\bar{x}^{\cdot},\quad \theta_1^{\cdot} = 0,\quad A_2^{\cdot} = \frac{\varepsilon}{2}N_4A_2\bar{x}^{\cdot},\quad \theta_2^{\cdot} = 0 \tag{3.6}$$
$$B\bar{x}^{\cdot\cdot} = \frac{\varepsilon}{2}[A_1^2\Delta_1 + A_2^2\Delta_2],\quad \Delta_1 = K_2 + h_1b^2,\quad \Delta_2 = K_4 - h_1d^2$$

Using the relation $\psi^{\cdot} = x^{\cdot} = \varepsilon x^{\cdot}$ we can integrate a system (3.6) to obtain (A_1°, A_2°, θ_1° and θ_2° are constants)

$$A_1 = A_1^{\circ} \exp \frac{N_1\psi}{2}, \quad \theta_1 = \theta_1^{\circ}, \quad A_2 = A_2^{\circ} \exp \frac{N_4\psi}{2}, \quad \theta_2 = \theta_2^{\circ}$$

Let us assume that at the initial instant $\psi = 0$. The dependence of ψ on time is given with the accuracy of up to the quadrature by

$$B\psi^{\cdot 2} = B\psi_0^{\cdot 2} + \varepsilon^2 \left\{ A_1^{\circ 2} \frac{\Delta_1}{N_1} [\exp(N_1\psi) - 1] + A_2^{\circ 2} \frac{\Delta_2}{N_4} [\exp(N_4\psi) - 1] \right\} \tag{3.7}$$

where $\psi_0^{\cdot}$ is the value of $\psi^{\cdot}$ at the initial instant. In what follows, we shall assume that $\theta_1^{\circ} = \theta_2^{\circ} = 0$, and thus obtain an approximate solution to the system of differential equations (3.4) in the form

$$Y = \varepsilon A_1^{\circ} \left[\exp \frac{N_1\psi}{2} \right] \cos bt, \quad Z = \varepsilon A_2^{\circ} \left[\exp \frac{N_4\psi}{2} \right] \cos dt \tag{3.8}$$

where ψ is found from (3.7).

Equations (3.8) describe the transversal oscillations of the body S. Before analysing the process of motion of S, we shall represent the constants $\Delta_1, \Delta_2, N_1, N_4$ as functions of b^2, d^2 and γ. The matrix relation

$$E = T(\gamma) \begin{Vmatrix} b^2 & 0 \\ 0 & d^2 \end{Vmatrix} T(-\gamma)$$

enables us to express $P/\Delta, Q/\Delta$ and R/Δ in terms of b^2, d^2 and γ. We obtain

$$\frac{P}{\Delta} = a + \frac{\bar{A}}{mg} (b^2 \cos^2 \gamma + d^2 \sin^2 \gamma),$$

$$\frac{Q}{\Delta} = \frac{\sqrt{\bar{A}\bar{C}}}{mg} (b^2 - d^2) \sin \gamma \cos \gamma$$

$$\frac{R}{\Delta} = a + \frac{\bar{C}}{mg} (b^2 \sin^2 \gamma + d^2 \cos^2 \gamma)$$

By virtue of the assumption that $\rho_2 > \rho_1, 0 < \alpha < \pi/2, b > d$, we can choose $0 < \gamma < \pi/2$ to obtain

$$N_1 = \frac{a}{g} \frac{\sin \gamma \cos \gamma}{\sqrt{\bar{A}\bar{C}}} b^2 (\bar{C} - \bar{A}), \quad \Delta_1 = -b^2 N_1$$

$$N_4 = \frac{a}{g} \frac{\sin \gamma \cos \gamma}{\sqrt{\bar{A}\bar{C}}} d^2 (\bar{A} - \bar{C}), \quad \Delta_2 = -d^2 N_4$$

The assumption that $\bar{A} > \bar{C}$ implies that $N_1 < 0$ and $N_4 > 0$. We write the equation (3.7) used for determining ψ, in the form

$$B\psi^{\cdot 2} = H(\psi), \; H(\psi) = B\psi_0^{\cdot 2} + b^2 B_1^2 [1 - \exp(N_1\psi)] + d^2 B_2^2 [1 - \exp(N_4\psi)] \tag{3.9}$$

The quantities $B_1 = \varepsilon A_1^{\circ}$ and $B_2 = \varepsilon A_2^{\circ}$ characterize the initial amplitudes of two forms of transverse oscillations of S.

Analysis of the function $H(\psi)$ shows that it vanishes at two values ψ_1 and ψ_2, of the coordinate ψ, with different signatures. For this reason the motion represents, a periodic oscillatory motion in ψ, between two values ψ_1 and ψ_2, for which the angular velocity $\psi^{\cdot}$ of the body S becomes zero.

Let the body S be stationary at the initial instant. We impart to the point of contact I of the body with the plane an infinitesimal displacement. The coordinates (ξ_0, ζ_0) of the point I relative to the axes Gx_1, Gx_3 are infinitesimal and of the order of ε. Retaining in the system terms of up to the second order of smallness in ε, we obtain the values of initial perturbations u_0, and v_0 for u and v, coorresponding to the displacement of the point

$$u_0 = Q\xi_0 + R\zeta_0, \; v_0 = -P\xi_0 - Q\zeta_0$$

The initial amplitudes of two forms of transversal oscillations of the body S are obtained in terms of u_0 and v_0 using the relations

$$B_1 = \sqrt{\bar{A}} u_0 \cos \gamma + \sqrt{\bar{C}} v_0 \sin \gamma, \; B_2 = -\sqrt{\bar{A}} u_0 \sin \gamma + \sqrt{\bar{C}} v_0 \cos \gamma$$

The relation

$$B\psi_0^{\cdot\cdot} = \frac{a}{2g}\frac{\sin\gamma\cos\gamma}{\sqrt{\bar{A}\bar{C}}}(\bar{A}-\bar{C})\chi, \quad \chi = b^4B_1^2 - d^4B_2^2$$

shows that if $\chi > 0$, then at the beginning of the motion $\psi^{\cdot}$ and ψ are both positive, the transversal oscillations in Y decay, and the body rotates in the positive direction. On the contrary, if $\chi < 0, \psi^{\cdot}$ and ψ are negative, transversal oscillations in Z decay and the body rotates in the negative direction.

4. Application to a particular model of the celtic stone. The study carried out above can be applied to the model of celtic stone used in /1/, with the following numerical values of the parameters $m = 0.15$ kg, $A = 4.5\cdot10^{-4}$ kg.m^2, $B = 6\cdot10^{-4}$ kg.m^2, $C = 2\cdot10^{-4}$ kg.m^2, $a = 0.01$ m, $\rho_1 = 0.025$ m, $\rho_2 = 0.5$ m. The angle α can vary from 0 to $\pi/2$, and ω is arbitrary. The inequalities (2.5) hold and the position of equilibrium (1.1) is stable. Since the inequality (2.8) and the first inequality of (2.7) both hold, the stability of the motions (1.1) is guaranteed for $\omega > \omega_0 \simeq 32$ rad/s. If $\omega < \omega_0$, then we have instability. The analysis carried out in Sect.3 have shown that the direction of rotation resulting from an infinitesimal displacement of the point of contact depends on the sign of χ. If the displacement of the point of contact is caused by an impact on the upper part of the top, then it can be shown that the direction of the resulting rotation depends on the position of the point at which the impact is delivered. The inequality $\chi > 0$ expressing the condition that the body S rotates in the positive direction can be written in the form

$$[b_1 - kb_2 + (c_1 - kc_2)\,\mathrm{ctg}\,\Phi][b_1 + kb_2 + (c_1 + kc_2)\,\mathrm{ctg}\,\Phi] > 0 \tag{4.1}$$

$$b_1 = \sqrt{\bar{A}}Q\cos\gamma - \sqrt{\bar{C}}P\sin\gamma, \quad c_1 = \sqrt{\bar{A}}R\cos\gamma - \sqrt{\bar{C}}Q\sin\gamma$$

$$b_2 = -\sqrt{\bar{A}}Q\sin\gamma - \sqrt{\bar{C}}P\cos\gamma, \quad c_2 = -\sqrt{\bar{A}}R\sin\gamma - \sqrt{\bar{C}}Q\cos\gamma$$

$$k = d^2/b^2$$

The angle Φ determines the initial position of the point of contact I of the body with the plane relative to the axes Gx_1 and Gx_3, by means of the relations

$$\xi_0 = \rho_0\sin\Phi, \quad \zeta_0 = \rho_0\cos\Phi, \quad \rho_0 = (\xi_0^2 + \zeta_0^2)^{1/2}, \quad 0 < \Phi < \pi$$

The inequality (4.1) makes it possible to determine the values of Φ for which the body S rotates in the positive direction. The limiting values Φ_1 and Φ_2 of the angle Φ at which the left-hand part of (4.1) vanishes, are obtained from the equations

$$\mathrm{ctg}\,\Phi_1 = \frac{kb_2 - b_1}{c_1 - kc_2}, \quad 0 < \Phi_1 < \pi; \quad \mathrm{ctg}\,\Phi_2 = -\frac{kb_2 + b_1}{kc_2 + c_1}, \quad 0 < \Phi_2 < \pi$$

Numerical analysis shows that Φ_2 vanishes when $\alpha = \alpha_0 \simeq 55°$. If $0 < \alpha \leqslant \alpha_0$, then if the initial position of the point of contact is such that S rotates in the positive direction, then Φ satisfies the inequalities $0 < \Phi < \Phi_2$ and $\Phi_1 < \Phi < \pi$. If $\alpha_0 < \alpha < \pi/2$, then we have, for this initial position, $\Phi_1 < \Phi < \Phi_2$. Numerical analysis indicates that if, e.g. we vary α from 0 to 60° then Φ_1 decreases, while Φ_2 decreases over two intervals, $[0, \alpha_0]$ and $[\alpha_0, 60°]$.

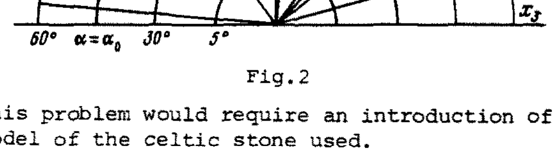

Fig.2

Fig.2 depicts, for various values of α, the zones, in which the point of contact should appear in order for the body S to rotate in the positive direction. In every of these cases the conclusion reached is that arrived at in /1/, namely that there exists a preferred direction of rotation. Indeed, the zones corresponding to rotation in positive direction are wider than those corresponding to rotations in the negative direction.

The problem of change in the direction of rotation in case when the initial velocity is not zero, has not been dealt with here. The analysis of equations of perturbed motion is considerably more complex than that of the equilibrium case. Solution of this problem would require an introduction of some simplifying assumptions concerning the model of the celtic stone used.

REFERENCES

1. LINDBERG R.E. and LONGMAN R.W., On the dynamic behaviour of the Wobblestone. AAS/AIAA. Astrodynamic Specialist Conference. Lake Tahoe, Nevada, 1981.
2. WALKER G.T., On a dynamic top. J. Pure and Appl. Math. Vol.28, 1896.
3. MAGNUS K., Die Stabilität der Drehbewegungen eines unsymmetrischen Körpers auf horizontaler Unterlage. Aus Theorie und Praxis der Ingenieurwissenschaften. Berlin. M., 1971.
4. MAGNUS K., Zur Theorie der Keltischen Wackelsteine. ZAMM, B.54, No.4, 1974.
5. RUMIANTSEV V.V., On the stability of rotation of a heavy gyrostat on a horizontal plane. Izv. Akad. Nauk SSSR, MTT, No.4, 1980.
6. KARAPETIAN A.V., On the problem of steady motions stability of nonholonomic systems. PMM Vol.44, No.3, 1980.
7. KARAPETIAN A.V., On permanent rotations of a heavy solid body on a absolutely rough horizontal plane. PMM Vol.45, No.5, 1981.
8. ASTAPOV I.S., On the stability of rotation of a celtic stone. Vestn. MGU, Ser. Matem. mekhan. No.2, 1980.
9. WALKER J., The Amateur Scientist. The Mysterious rattleback. Scient. Amer. Vol.241, No.4, 1979.
10. CAUGHEY T.K., A mathematical model of the "rattleback". Int. J. Nonlinear Mechanics. Vol. 15, No.4, 1980.
11. KARAPETIAN A.V., On the stability of steady motions of Chaplygin's nonholonomic systems. PMM Vol.42, No.5, 1978.
12. MALKIN I.G., Theory of Stability of Motion. Moscow, NAUKA, 1966.
13. ROSEAU M., Equations Différentielles. Paris. Masson, 1978.

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